ASYMPTOTIC FREENESS OF RANDOM PERMUTATION MATRICES WITH RESTRICTED CYCLE LENGTHS

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ABSTRACT. Let $A_1, A_2, \ldots, A_s$ be a finite sequence of (not necessarily disjoint, or even distinct) non-empty sets of positive integers such that each A_r either is a finite set or satisfies $\sum_{j \in \mathbb{N} \setminus A_r} \frac{1}{j} < \infty$. It is shown that an independent family $U_1, U_2, \ldots, U_s$ of uniformly distributed random $N \times N$ permutation matrices with cycle lengths restricted to $A_1, A_2, \ldots, A_s$, respectively, converges in $*$-distribution as $N \to \infty$ to a $*$-free family $u_1, u_2, \ldots, u_s$ of non-commutative random variables, where each u_r is a $(\max A_r)$-Haar unitary (if A_r is a finite set) or a Haar unitary (if A_r is an infinite set). Under the additional assumption that each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite, it is shown that the convergence in $*$-distribution actually holds almost surely.

1. INTRODUCTION.

Almost from the initial development of free probability theory, it was shown by Voiculescu that, in many important cases, independent families of $N \times N$ random matrices are asymptotically free in the limit $N \to \infty$; as shown in [8], this asymptotic freeness phenomenon occurs for square matrices with independent complex Gaussian entries and for random unitary matrices.

Another instance of asymptotic freeness which arises in a natural situation is the case of an independent family of uniformly distributed $N \times N$ random permutation matrices, which was shown in [5] to provide an asymptotic model as $N \to \infty$ for a $*$-free family of Haar unitaries in a non-commutative probability space. It was shown in [4] that these matrices are also asymptotically free from various types of independent families of complex Gaussian matrices.

Closely related to the distribution of a Haar unitary is that of a d-Haar unitary for a positive integer d; a d-Haar unitary in a $*$-probability space $(\mathcal{A}, \varphi)$ is a unitary $u \in \mathcal{A}$ satisfying $u^d = 1$ and $\varphi(u^k) = 0$ if k is not a multiple of d. It is immediate that this distribution is the same as that of an $N \times N$ random permutation matrix which is uniformly distributed over the set

$$\mathcal{S}_N^{(d)} := \{\sigma \in \mathcal{S}_N \mid \text{all cycles of } \sigma \text{ have length } d\} \tag{1.1}$$

(with N being a multiple of d, so that the set $\mathcal{S}_N^{(d)}$ is non-empty). A natural question is then whether asymptotic freeness still occurs for an independent family $U_1, U_2, \ldots, U_s$, where each U_r is an $N \times N$ random permutation matrix which is uniformly distributed over either $\mathcal{S}_N$ or $\mathcal{S}_N^{(d_r)}$ for some positive integer d_r (with N chosen such that each of the sets $\mathcal{S}_N^{(d_r)}$ is non-empty). This paper gives an affirmative answer to a more general form of this question, and provides a unified framework for the two cases in which U_r is uniformly distributed over $\mathcal{S}_N$ or $\mathcal{S}_N^{(d_r)}$.

More concretely, for every non-empty set A of positive integers, let

$$\mathcal{S}_N^{(A)} := \{\sigma \in \mathcal{S}_N \mid \text{the length of every cycle of } \sigma \text{ belongs to } A\}; \tag{1.2}$$

the cases in which A consists either of all positive integers or of a single positive integer were mentioned above. Let $u_1, u_2, \ldots, u_s$ be a $*$-free family of random variables in a $*$-probability space, with each u_r being either a Haar unitary or a d_r-Haar unitary for some positive integer d_r. Then the question asked in the previous paragraph is whether an asymptotic model for the family $u_1, u_2, \ldots, u_s$ is given by an independent family $U_1, U_2, \ldots, U_s$, where each U_r is an $N \times N$ random permutation matrix which is uniformly distributed over $\mathcal{S}_N^{(A_r)}$, with

$$A_r := \begin{cases} \{d_r\} & \text{if } u_r \text{ is a } d_r\text{-Haar unitary} \\ \mathbb{N} & \text{if } u_r \text{ is a Haar unitary} \end{cases} \tag{1.3}$$

(and N chosen such that each of the sets $\mathcal{S}_N^{(A_r)}$ is non-empty). The main result of this paper (Theorem 4.6 below) is that the above statement is true and that, in fact, it can be considerably strengthened by allowing

$$A_r = \begin{cases} \text{any finite set of positive integers with } d_r = \max A_r & \text{if } u_r \text{ is a } d_r\text{-Haar unitary} \\ \text{any infinite set of positive integers with } \sum\limits_{j \in \mathbb{N} \setminus A_r} \frac{1}{j} < \infty & \text{if } u_r \text{ is a Haar unitary.} \end{cases}$$
$$\tag{1.4}$$

Under the additional assumption that each of the sets A_r either consists of a single positive integer or is infinite, it is shown that the convergence in $*$-distribution of the family $U_1, U_2, \ldots, U_s$ actually holds almost surely.

Moreover, the same techniques used in the proof of Theorem 3.1 of [4] can be applied to show that the family $U_1, U_2, \ldots, U_s$ is asymptotically $*$-free from various types of independent families of complex Gaussian matrices, with the convergence in $*$-distribution holding almost surely under the same additional assumption as in the previous paragraph (see 4.10 below).

The proof of the main theorem of the paper makes use of two principal ideas, which are explored in Sections 2 and 3. The first involves the analysis of a certain class of directed, edge-colored graphs and their congruences, which were introduced in [6]. Section 2 of the paper is devoted to the study of the relevant facts about these graphs, and an important ingredient is a quantity which bears an uncanny resemblance to the classical Euler characteristic. Because of this resemblance, I have called this quantity the *loop-characteristic* of a graph (see 2.9 below); as far as I have been able to determine, it does not appear in the graph theory literature.

Section 3 of the paper is devoted to the evaluation of certain probabilities related to random permutations which are uniformly distributed over $\mathcal{S}_N^{(A)}$. More specifically, for a fixed non-empty set A of positive integers and for a fixed $k \in A$, the relevant concept for the present purposes is the asymptotic behavior as $N \to \infty$ (with $\mathcal{S}_N^{(A)} \neq \emptyset$) of the expected number of cycles of length k of such a random permutation. If A is a finite set, this asymptotic behavior can be computed with a moderate amount of effort by using Hayman's method (a saddle point method) for determining the asymptotic behavior of power series coefficients of certain complex analytic functions; here Hayman's method is applicable to the exponential generating function for the number of permutations in $\mathcal{S}_N^{(A)}$. If A is an infinite set, it turns out that this

asymptotic behavior has connections to recent research (see [2] and [3]), which shows that it can be computed under the assumption that A satisfies the condition appearing in the second part of (1.4).

Finally, Section 4 of the paper sets the free probabilistic framework which is necessary for the formulation of the main result, which is then stated and proved.

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2. Edge-Colored Graphs and Their Congruences.

This section discusses a class of graphs which is needed in the sequel. These graphs were introduced in [6], which contains additional information about their congruences and homomorphisms. With slight modifications, the definitions and notation appearing in 2.3, 2.4, 2.5, and 2.7 below are taken from that work.

2.1. Notation. (a) Throughout the paper, s is a fixed positive integer and $d_1, d_2, \dots, d_s$ are fixed elements of the set $\{1, 2, \dots\} \cup \{\infty\}$. If n is a positive integer, then $[n]$ denotes the set

$$[n] := \{1, 2, \dots, n\}. \tag{2.1}$$

(b) The free monoid generated by some fixed symbols $g_1, g_1^*, g_2, g_2^*, \dots, g_s, g_s^*$ is denoted by $\mathbb{F}$. The elements of $\mathbb{F}$ are "words" in $g_1, g_1^*, g_2, g_2^*, \dots, g_s, g_s^*$. The identity of $\mathbb{F}$ is the "empty word" and is denoted by e.

We shall denote by $\approx$ the congruence of $\mathbb{F}$ generated by the relations

$$\{g_r g_r^* \approx g_r^* g_r \approx e \mid r \in [s]\} \cup \{g_r^{d_r} \approx (g_r^*)^{d_r} \approx e \mid r \in [s]; d_r < \infty\}. \tag{2.2}$$

In other words, $\approx$ is the smallest equivalence relation on $\mathbb{F}$ containing the above relations and having the property that $w_1 \approx w_2 \implies u w_1 v \approx u w_2 v$ for every $u, v, w_1, w_2 \in \mathbb{F}$. It is clear that for every $w \in \mathbb{F} \setminus \{e\}$, we have that $w \approx e$ iff some cyclic permutation of the symbols of w is of the form vx, where $v \approx e$ and

$$x \in \{g_r g_r^*, g_r^* g_r \mid r \in [s]\} \cup \{g_r^{d_r}, (g_r^*)^{d_r} \mid r \in [s]; d_r < \infty\}. \tag{2.3}$$

2.2. Partitions. Let V be a non-empty set.

(a) The set of all partitions of V is denoted by $\Pi(V)$. If $\pi \in \Pi(V)$ and $a \in V$, then $\pi(a)$ denotes the block of π to which a belongs, $|\pi|$ denotes the number of blocks of π, and $\overset{\pi}{\sim}$ denotes the equivalence relation on V associated to π:

$$a \overset{\pi}{\sim} b \overset{\text{def}}{\iff} \pi(a) = \pi(b) \qquad \forall a, b \in V. \tag{2.4}$$

If $\pi \in \Pi(V)$ and W is a non-empty subset of V, then the *restriction* of π to W is the partition $\rho \in \Pi(W)$ defined by

$$a \overset{\rho}{\sim} b \overset{\text{def}}{\iff} a \overset{\pi}{\sim} b \qquad \forall a, b \in W. \tag{2.5}$$

(b) For every $\pi, \rho \in \Pi(V)$, we write $\pi \leq \rho$ iff every block of π is completely contained in a block of ρ. This gives a partial order which makes $\Pi(V)$ into a complete lattice, the meet operation of which is denoted by $\wedge$. If $\pi, \rho \in \Pi(V)$ and $\pi \leq \rho$, then ρ/π denotes the partition of π defined by

$$\pi(a) \overset{\rho/\pi}{\sim} \pi(b) \overset{\text{def}}{\iff} a \overset{\rho}{\sim} b \quad \forall a, b \in V. \tag{2.6}$$

2.3. Graphs. (a) Throughout the paper, *graph* means a finite, directed, edge-colored graph with colors from the set $[s]$. Formally, a graph Γ consists of a non-empty finite *vertex set* V together with (not necessarily disjoint) subsets $E_1, E_2, \dots, E_s$ of $V \times V$; for every $r \in [s]$ and for every $a, b \in V$, we shall say that $a \to b$ is an *r-edge* (or an *edge of color r*) of Γ iff $(a, b) \in E_r$. A graph is said to be *trivial* iff it has no edges (of any color).

A graph is said to be *monocolored* iff all its edges have the same color. For every $r \in [s]$, the (monocolored) graph obtained from a graph Γ by removing all edges which are not of color r is denoted by $\Gamma(r)$.

(b) A *subgraph* of a graph Γ is a graph Γ' which has vertex set contained in that of Γ and which has the property that for every $r \in [s]$, each r-edge of Γ' is an r-edge of Γ.

A graph Γ is said to be *connected* iff for every two distinct vertices a and b of Γ, there exist a positive integer n and vertices $a_0, a_1, \dots, a_n$ of Γ with $a_0 = a$ and $a_n = b$ such that for every $k \in [n]$, there exists an edge (of any color and either direction) between a_{k-1} and a_k. A *connected component* of a graph Γ is a maximal connected subgraph of Γ.

(c) Two graphs Γ_1 and Γ_2 with vertex sets V_1 and V_2, respectively, are said to *isomorphic*, written $\Gamma_1 \cong \Gamma_2$, iff there exists a bijection $\Phi : V_1 \to V_2$ (called an *isomorphism*) such that for every $r \in [s]$ and for every $a, b \in V_1$, we have that

$$\Phi(a) \to \Phi(b) \text{ is an } r\text{-edge of } \Gamma_2 \iff a \to b \text{ is an } r\text{-edge of } \Gamma_1.$$

2.4. Word-graphs. Let $\mathbb{F}$ be as in 2.1(b). For every $w := g_{r_1}^{\varepsilon_1} g_{r_2}^{\varepsilon_2} \cdots g_{r_n}^{\varepsilon_n} \in \mathbb{F} \setminus \{e\}$, where n is a positive integer, $r_1, r_2, \dots, r_n \in [s]$, and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{1, *\}$, the *word-graph* Γ_w associated to w is the graph with vertex set $[n]$ and with n edges, described as follows: for every $k \in [n]$, the k-th edge of Γ_w has color r_k and is

$$\begin{cases} k \xleftarrow{r_k} k+1 & \text{if } \varepsilon_k = 1 \\ k \xrightarrow{r_k} k+1 & \text{if } \varepsilon_k = * \end{cases} \tag{2.7}$$

(with the convention that $n + 1 := 1$). Figure 1 below shows an example of a word-graph.

2.5. Quotient graphs. For every graph Γ and for every partition π of the vertex set of Γ, the *quotient* of Γ by π is the graph Γ/π which has vertex set π and which is defined by requiring that for every $r \in [s]$ and for every pair of vertices a and b of Γ, we have that $\pi(a) \to \pi(b)$ is an r-edge of Γ/π iff there exists an r-edge $a' \to b'$ of Γ with $a' \overset{\pi}{\sim} a$ and $b' \overset{\pi}{\sim} b$. Figure 2 below shows an example of a quotient graph.

It is immediate that any quotient of a connected graph is connected; in particular, with the notation from 2.4, we have that Γ_w/π is connected for every $w \in \mathbb{F} \setminus \{e\}$ and for every partition π of the vertex set of Γ_w.
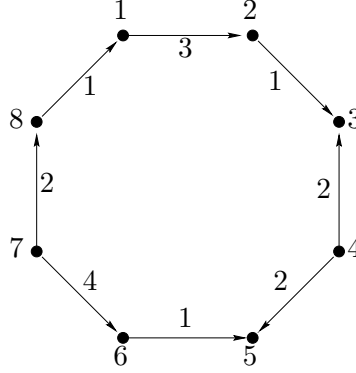
FIGURE 1.   The word-graph associated to the word $g_3^* g_1^* g_2 g_2^* g_1 g_4 g_2^* g_1^*$.
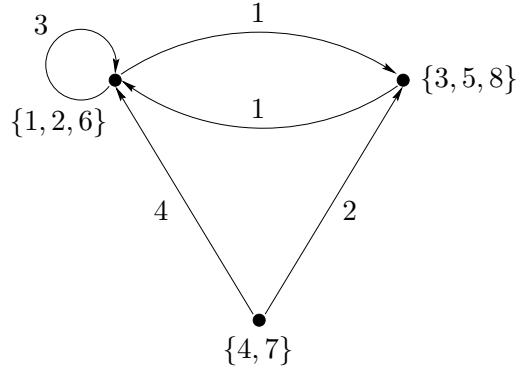
FIGURE 2.   The quotient of the graph from Figure 1 by the partition $\{\{1,2,6\},\{3,5,8\},\{4,7\}\}$.

2.6. Lemma (Second Isomorphism Theorem). *If Γ is a graph with vertex set V and $\pi, \rho \in \Pi(V)$ are such that $\pi \leq \rho$, then $\Gamma/\rho \cong (\Gamma/\pi)/(\rho/\pi)$, where ρ/π is as in 2.2(b).*

Proof. Let $\Phi : \rho \to \rho/\pi$ be defined by

$$\Phi\big(\rho(a)\big) = (\rho/\pi)\big(\pi(a)\big) \qquad \forall a \in V.$$

From the definition of ρ/π, it is immediate that Φ is a well-defined bijection. For every $r \in [s]$ and for every $a, b \in V$, we have that

$(\rho/\pi)\big(\pi(a)\big) \to (\rho/\pi)\big(\pi(b)\big)$is an r-edge of $(\Gamma/\pi)/(\rho/\pi)$

$\iff$ there exists an r-edge $A \to B$ of Γ/π such that $A \overset{\rho/\pi}{\sim} \pi(a)$ and $B \overset{\rho/\pi}{\sim} \pi(b)$

$\iff$ there exists an r-edge $a' \to b'$ of Γ such that $\pi(a') \overset{\rho/\pi}{\sim} \pi(a)$ and $\pi(b') \overset{\rho/\pi}{\sim} \pi(b)$

$\iff$ there exists an r-edge $a' \to b'$ of Γ such that $a' \overset{\rho}{\sim} a$ and $b' \overset{\rho}{\sim} b$

$\iff$ $\rho(a) \to \rho(b)$ is an r-edge of Γ/ρ.

Hence Φ is an isomorphism between Γ/ρ and $(\Gamma/\pi)/(\rho/\pi)$. $\square$

2.7. Admissible graphs and congruences. (a) A graph Γ is said to be *admissible* iff no two distinct edges of Γ having the same color begin at the same vertex or end at the same vertex.

(b) A *congruence* of a graph Γ is a partition π of the vertex set of Γ such that for every pair $a_1 \to b_1$ and $a_2 \to b_2$ of edges of Γ having the same color, we have that $a_1 \overset{\pi}{\sim} a_2 \iff b_1 \overset{\pi}{\sim} b_2$. The set of all congruences of Γ is denoted by $\mathrm{Con}(\Gamma)$. It is easily seen that

$$\pi \wedge \rho \in \mathrm{Con}(\Gamma) \qquad \forall \pi, \rho \in \mathrm{Con}(\Gamma) \tag{2.8}$$

and that for every partition π of the vertex set of Γ, we have that

$$\Gamma/\pi \text{ is admissible} \iff \pi \in \mathrm{Con}(\Gamma). \tag{2.9}$$

(c) Let Γ be an admissible graph. For every $r \in [s]$, an *r-path* (or a *path of color r*) of Γ is a non-trivial connected component of $\Gamma(r)$ (notation and terminology as in 2.3). The *length* $l(P)$ of a path P (of any color) of Γ is the number of edges of P. The graph obtained from Γ by removing all edges of a path P is denoted by $\Gamma \setminus P$.

We shall make use of the following observation.

2.8. Lemma. *If Γ is a non-trivial connected admissible graph, then there exists a path P of Γ such that $\Gamma \setminus P$ has at most one non-trivial connected component.*

Proof. The proof is by induction on the number of paths of Γ. If Γ consists of a single path P, then it is clear that $\Gamma \setminus P$ has no non-trivial connected components.

Now suppose that the assertion of the lemma holds for every non-trivial connected admissible graph with number of paths less than or equal to some positive integer n, and let Γ be a non-trivial connected admissible graph with $n + 1$ paths. Let Γ' be a subgraph of Γ with minimal number of paths relative to the following property: there exists a path Q of Γ such that Γ' is a non-trivial connected component of $\Gamma \setminus Q$.

If Γ' consists of a single path P, then it is clear that $\Gamma \setminus P$ has exactly one non-trivial connected component. Finally, suppose that Γ' has at least two paths. By the induction hypothesis, there exists a path P of Γ' such that $\Gamma' \setminus P$ has exactly one non-trivial connected component Γ''. If Γ'' has no vertices in common with Q, then Γ'' is a non-trivial connected component of $\Gamma \setminus P$, which contradicts the minimality of the number of paths of Γ'. Thus Γ'' has at least one vertex in common with Q, from which it follows that $\Gamma \setminus P$ has exactly one non-trivial connected component. $\square$

2.9. The loop-characteristic of admissible graphs. Let Γ be an admissible graph. It is clear that every path of Γ is a circuit which is not self-intersecting. For every $r \in [s]$, an *r-loop* (or a *loop of color r*) of Γ is an r-path of Γ which is a closed circuit, and an *r-string* (or a *string of color r*) of Γ is an r-path of Γ which is not a loop. The *loop-characteristic* of Γ is the integer $\chi(\Gamma)$ defined by

$$\chi(\Gamma) = V(\Gamma) - E(\Gamma) + L(\Gamma), \tag{2.10}$$

where $V(\Gamma)$, $E(\Gamma)$, and $L(\Gamma)$ denote the numbers of vertices, edges, and loops (of any color) of Γ, respectively. For instance, the (admissible) graph shown in Figure 2 above has a 1-loop of length 2, a 2-string of length 1, a 3-loop of length 1, a 4-string of length 1, and has loop-characteristic $3 - 5 + 2 = 0$.

2.10. Proposition. *If* Γ *is a connected admissible graph, then* $\chi(\Gamma) \leq 1$, *with equality holding iff each path* P *of* Γ *has the following property: every connected component* C *of* $\Gamma \setminus P$ *has exactly one vertex in common with* P *and satisfies* $\chi(C) = 1$. *Moreover, if some path of* Γ *has this property, then* $\chi(\Gamma) = 1$.

Proof. The proof is by induction on the number of paths of Γ. If Γ has no paths (and therefore no edges and exactly one vertex), then it is immediate that $\chi(\Gamma) = 1$.

Now suppose that the assertion of the proposition holds for every connected admissible graph with number of paths less than some positive integer n, let Γ be a connected admissible graph with n paths, and let P be any path of Γ. Let $C_1, C_2, \dots, C_k$ be the connected components of $\Gamma \setminus P$ and let n_i be the number of vertices which P has in common with each C_i. Then

$$
\begin{aligned}
\chi(\Gamma) &= \chi(P) + \chi(C_1) + \chi(C_2) + \cdots + \chi(C_k) - (n_1 + n_2 + \cdots + n_k) \quad \text{[by the definition of } \chi] \\
&\leq 1 + k - (n_1 + n_2 + \cdots + n_k) \qquad \text{[since } \chi(P) = 1 \text{ and by the induction hypothesis]} \\
&\leq 1, \qquad\qquad\qquad\qquad\qquad \text{[since } \Gamma \text{ is connected, so } n_1, n_2, \dots, n_k \geq 1]
\end{aligned}
$$

with equalities holding iff $\chi(C_1) = \chi(C_2) = \cdots = \chi(C_k) = 1$ and $n_1 = n_2 = \cdots = n_k = 1$. $\square$

2.11. Strongly admissible graphs and strong congruences. (a) A graph Γ is said to be *strongly admissible* iff it is admissible and for every $r \in [s]$, each r-loop of Γ has length d_r and each r-string of Γ has length strictly less than d_r [notation as in 2.1(a)]. In particular, a strongly admissible graph cannot contain any r-loops for any $r \in [s]$ with $d_r = \infty$.

(b) A *strong congruence* of a graph Γ is a congruence π of Γ such that Γ/π is strongly admissible. The set of all strong congruences of Γ is denoted by $\mathrm{SCon}(\Gamma)$. It is easily seen that

$$
\pi \wedge \rho \in \mathrm{SCon}(\Gamma) \qquad \forall \pi, \rho \in \mathrm{SCon}(\Gamma). \tag{2.11}
$$

2.12. Remarks. Let Γ' be a subgraph of a graph Γ.

(a) If Γ is admissible (respectively, strongly admissible), then so is Γ'.

(b) If both Γ and Γ' are connected and admissible and $\chi(\Gamma) = 1$, then $\chi(\Gamma') = 1$.

Indeed, (a) is immediate from the definition of admissible and strongly admissible graphs, whereas (b) follows from Proposition 2.10, since each path P' of Γ' is a subgraph of some path P of Γ and every connected component of $\Gamma' \setminus P'$ is a subgraph of some connected component of $\Gamma \setminus P$.

2.13. Lemma. *If* Γ *is a connected strongly admissible graph and* $\pi \in \mathrm{SCon}(\Gamma)$ *is such that* $\chi(\Gamma/\pi) = 1$, *then all blocks of* π *are singletons.*

Proof. The proof is by induction on the number of paths of Γ. If Γ has no paths (and therefore no edges and exactly one vertex), then the assertion holds vacuously, since the only partition of the vertex set of Γ consists of a singleton block.

Now suppose that the assertion of the lemma holds for every connected strongly admissible graph with number of paths less than some positive integer n, let Γ be a connected strongly admissible graph with n paths, and let $\pi \in \mathrm{SCon}(\Gamma)$ be such that $\chi(\Gamma/\pi) = 1$. Let P be any path of Γ (of some color r). Since Γ is strongly admissible, P is either a loop of length d_r or a

string of length strictly less than d_r, which implies that no two distinct vertices of P belong to the same block of π (since in that case Γ/π would have an r-loop of length strictly less than d_r, contradicting the assumption that Γ/π is strongly admissible). Let a and b be two distinct vertices of Γ; it will be shown that $a \overset{\pi}{\nsim} b$.

If a and b belong to the same connected component C of $\Gamma \setminus P$, let ρ be the restriction of π to the vertices of C, as in 2.2(a). Since C is a connected subgraph of Γ and C/ρ is isomorphic to a connected subgraph of Γ/π, it follows from Remarks 2.12 and the induction hypothesis that all blocks of ρ are singletons; in particular, $a \overset{\pi}{\nsim} b$.

Finally, suppose that a and b belong to distinct connected components C_1 and C_2 of $\Gamma \setminus P$. Since Γ is connected, there exist vertices c and d of P such that c belongs to C_1 and d belongs to C_2. Let $\overline{P}$ be the r-path of Γ/π containing the vertices $\pi(c)$ and $\pi(d)$, and let $\overline{C_1}$ and $\overline{C_2}$ be the connected components of $(\Gamma/\pi) \setminus \overline{P}$ containing the vertices $\pi(a)$ and $\pi(b)$, respectively. Since $\chi(\Gamma/\pi) = 1$, it follows from Proposition 2.10 that $\overline{P}$ has exactly one vertex in common with $\overline{C_1}$ [namely $\pi(c)$] and with $\overline{C_2}$ [namely $\pi(d)$]. Since $\pi(c) \neq \pi(d)$ (as c and d are distinct vertices of P), it follows that $\overline{C_1} \neq \overline{C_2}$. In particular, $\pi(a) \neq \pi(b)$, so $a \overset{\pi}{\nsim} b$.   □

2.14. Proposition. *For every connected graph Γ, there exists at most one $\pi \in \mathrm{SCon}(\Gamma)$ such that $\chi(\Gamma/\pi) = 1$.*

Proof. Let $\pi, \rho \in \mathrm{SCon}(\Gamma)$ be such that $\chi(\Gamma/\pi) = \chi(\Gamma/\rho) = 1$. Then $\pi \wedge \rho \in \mathrm{SCon}(\Gamma)$ and $\Gamma/\pi \cong (\Gamma/\pi \wedge \rho)/(\pi/\pi \wedge \rho)$ by Lemma 2.6, so all blocks of $\pi/\pi \wedge \rho$ are singletons by Lemma 2.13, which implies that $\pi \leq \rho$. A similar argument shows that $\rho \leq \pi$, so $\pi = \rho$.   □

2.15. Lemma. *Let $\mathbb{F}$ be as in 2.1(b), let $v \in \mathbb{F} \setminus \{e\}$, and let x be as in (2.3). Then:*

(a) $\mathrm{card}\{\pi \in \mathrm{SCon}(\Gamma_x) \,|\, \chi(\Gamma_x/\pi) = 1\} = 1$.

(b) $\mathrm{card}\{\pi \in \mathrm{SCon}(\Gamma_{vx}) \,|\, \chi(\Gamma_{vx}/\pi) = 1\} = \mathrm{card}\{\rho \in \mathrm{SCon}(\Gamma_v) \,|\, \chi(\Gamma_v/\rho) = 1\}$.

Proof. Let $r \in [s]$ be such that $x \in \left\{ g_r g_r^*, g_r^* g_r, g_r^{d_r}, (g_r^*)^{d_r} \right\}$.

(a) By Proposition 2.14, it suffices to show that there exists $\pi \in \mathrm{SCon}(\Gamma_x)$ such that $\chi(\Gamma_x/\pi) = 1$. If $x = g_r g_r^*$ or $g_r^* g_r$ and $d_r = 1$, let $\pi = \{\{1, 2\}\}$. If $x = g_r g_r^*$ or $g_r^* g_r$ and $d_r > 1$, let $\pi = \{\{1\}, \{2\}\}$. Finally, if $d_r < \infty$ and $x = g_r^{d_r}$ or $(g_r^*)^{d_r}$, let $\pi = \{\{1\}, \{2\}, \dots, \{d_r\}\}$. In any of the three cases, it is easily seen that $\pi \in \mathrm{SCon}(\Gamma_x)$ and that $\chi(\Gamma_x/\pi) = 1$.

(b) By Proposition 2.14, it suffices to show that there exists $\pi \in \mathrm{SCon}(\Gamma_{vx})$ such that $\chi(\Gamma_{vx}/\pi) = 1$ iff there exists $\rho \in \mathrm{SCon}(\Gamma_v)$ such that $\chi(\Gamma_v/\rho) = 1$. Let n be the length of v.

($\Longrightarrow$) Suppose that there exists $\pi \in \mathrm{SCon}(\Gamma_{vx})$ such that $\chi(\Gamma_{vx}/\pi) = 1$, and let ρ be the restriction of π to $[n]$, as in 2.2(a).

First note that $1 \overset{\pi}{\sim} n + 1$; indeed, if $x = g_r g_r^*$ or $g_r^* g_r$, this is because π is a congruence, whereas, if $d_r < \infty$ and $x = g_r^{d_r}$ or $(g_r^*)^{d_r}$, it is because the r-path of Γ_{vx}/π which contains $\pi(1)$ must be a loop of length d_r. This implies that Γ_v/ρ is isomorphic to a connected subgraph of Γ_{vx}/π, so $\rho \in \mathrm{SCon}(\Gamma_v)$ and $\chi(\Gamma_v/\rho) = 1$ by Remarks 2.12.

($\Longleftarrow$) Suppose that there exists $\rho \in \mathrm{SCon}(\Gamma_v)$ such that $\chi(\Gamma_v/\rho) = 1$. The two cases in which $x = g_r g_r^*$ or $g_r^* g_r$, and in which $d_r < \infty$ and $x = g_r^{d_r}$ or $(g_r^*)^{d_r}$ are treated separately. In each of the two cases, there are several subcases to consider, depending on the structure of the graph Γ_v/ρ.

Case 1: $x = g_r g_r^*$ (the case when $x = g_r^* g_r$ is analogous).

If Γ_v/ρ has an r-edge $A \to \rho(1)$ for some block A of ρ, let π be the partition of $[n+2]$ which is obtained from ρ by adding $n+1$ to the block $\rho(1)$ and $n+2$ to the block A. Then Γ_{vx}/π is isomorphic to Γ_v/ρ, so $\pi \in \mathrm{SCon}(\Gamma_{vx})$ and $\chi(\Gamma_{vx}/\pi) = 1$.

If $d_r = 1$ and Γ_v/ρ has no r-path containing $\rho(1)$, let π be the partition of $[n+2]$ which is obtained from ρ by adding $n+1$ and $n+2$ to the block $\rho(1)$. Then Γ_{vx}/π is isomorphic to Γ_v/ρ with an extra r-loop of length 1 added at the vertex $\rho(1)$, so $\pi \in \mathrm{SCon}(\Gamma_{vx})$ and $\chi(\Gamma_{vx}/\pi) = 1$.

If $d_r > 1$ and Γ_v/ρ has no r-path containing $\rho(1)$, let π be the partition of $[n+2]$ which is obtained from ρ by adding $n+1$ to the block $\rho(1)$ and creating a singleton block $\{n+2\}$. Then Γ_{vx}/π is isomorphic to Γ_v/ρ with an extra r-string (of length 1) consisting of the edge $\{n+2\} \to \rho(1)$, so $\pi \in \mathrm{SCon}(\Gamma_{vx})$ and $\chi(\Gamma_{vx}/\pi) = 1$.

If $d_r > 1$ and Γ_v/ρ has an r-string P of the form $\rho(1) \to A_1 \to A_2 \to \cdots A_{d_r-1}$ (of length $d_r - 1$), let π be the partition of $[n+2]$ which is obtained from ρ by adding $n+1$ to the block $\rho(1)$ and $n+2$ to the block A_{d_r-1}. Then Γ_{vx}/π is isomorphic to Γ_v/ρ with an extra edge $A_{d_r-1} \to \rho(1)$ (which extends P to a loop of length d_r), so $\pi \in \mathrm{SCon}(\Gamma_{vx})$ and $\chi(\Gamma_{vx}/\pi) = 1$.

If $d_r > 2$ and Γ_v/ρ has an r-string P of the form $\rho(1) \to A_1 \to A_2 \to \cdots A_k$ (of length k) for some $1 \le k < d_r - 1$, let π be the partition of $[n+2]$ which is obtained from ρ by adding $n+1$ to the block $\rho(1)$ and creating a singleton block $\{n+2\}$. Then Γ_{vx}/π is isomorphic to Γ_v/ρ with an extra edge $\{n+2\} \to \rho(1)$ (which extends P to a string of length $k+1 < d_r$), so $\pi \in \mathrm{SCon}(\Gamma_{vx})$ and $\chi(\Gamma_{vx}/\pi) = 1$.

Case 2: $d_r < \infty$ and $x = g_r^{d_r}$ (the case when $x = (g_r^*)^{d_r}$ is analogous).

If Γ_v/ρ has no r-path containing $\rho(1)$, let π be the partition of $[n+d_r]$ which is obtained from ρ by adding $n+1$ to the block $\rho(1)$ and creating singleton blocks $\{n+2\}, \{n+3\}, \ldots \{n+d_r\}$. Then Γ_{vx}/π is isomorphic to Γ_v/ρ with an extra r-loop (of length d_r) which has no vertices in common with Γ_v/ρ other than $\rho(1)$, so $\pi \in \mathrm{SCon}(\Gamma_{vx})$ and $\chi(\Gamma_{vx}/\pi) = 1$.

If Γ_v/ρ has an r-path P of the form $A_j \to \cdots \to A_2 \to A_1 \to \rho(1) \to B_1 \to B_2 \to \cdots \to B_k$ (where at least one of j and k is non-zero, $j+k \le d_r$, and possibly $A_j = B_k$), let π be partition of $[n+d_r]$ which is obtained from ρ by adding $n+1$ to the block $\rho(1)$, $n+1+i$ to the block A_i for every $i \in [j]$, $n+d_r+1-i$ to the block B_i for every $i \in [k]$, and (if $j+k < d_r - 1$) creating singleton blocks $\{n+j+2\}, \{n+j+3\}, \ldots, \{n+d_r-k\}$. Then Γ_{vx}/π is isomorphic to Γ_v/ρ with the path P extended to an r-loop (of length d_r) which has no vertices in common with Γ_v/ρ other than the vertices of P, so $\pi \in \mathrm{SCon}(\Gamma_{vx})$ and $\chi(\Gamma_{vx}/\pi) = 1$. $\square$

2.16. Lemma. *Let $\mathbb{F}$ be as in in 2.1(b), let $w \in \mathbb{F} \setminus \{e\}$, and let $\pi \in \mathrm{SCon}(\Gamma_w)$. If Γ_w/π has a path P such that at most one vertex of P belongs to other paths of Γ_w/π, then some cyclic permutation of the symbols of w is of the form vx with x as in (2.3).*

Proof. It may be assumed that Γ_w is admissible (otherwise it has a mono-colored subgraph of the form $\bullet \to \bullet \leftarrow \bullet$ or $\bullet \leftarrow \bullet \to \bullet$, which immediately implies that some cyclic permutation of the symbols of w is of the form $vu_r^*u_r$ or $vu_ru_r^*$ for some $r \in [s]$). Let r be the color of the path P and let B be a vertex of P (that is, block of π) such that no other vertex of P belongs to other paths of Γ_w/π. Let Q be an r-path of Γ_w containing a vertex which belongs to B.

Suppose that Q has length less than d_r. Then Q is an r-string and the two endpoints of Q do not belong to the same block of π (else Γ_w/π contains an r-loop of length less than d_r, contradicting the assumption that it is strongly admissible). In particular, at least one of the two endpoints of Q does not belong to B, so the block of π containing this endpoint belongs to no path of Γ_w/π other than P. This implies that this endpoint belongs only to paths of Γ_w which have color r, which is plainly impossible by the definition of Γ_w.

Thus Q has length at least d_r, from which it is clear that $d_r < \infty$ and some cyclic permutation of the symbols of w is of the form $vu_r^{d_r}$ or $v\,(u_r^*)^{d_r}$. $\qquad\square$

2.17. Proposition. *If $\mathbb{F}$ is as in 2.1(b) and $w \in \mathbb{F} \setminus \{e\}$, then*

$$\mathrm{card}\big\{\pi \in \mathrm{SCon}(\Gamma_w) \,\big|\, \chi(\Gamma_w/\pi) = 1\big\} = \begin{cases} 1 & \text{if } w \approx e \\ 0 & \text{if } w \not\approx e. \end{cases} \qquad (2.12)$$

Proof. The proof is by induction on the length of w. If w has length 1, then $w = g_r$ or g_r^* for some $r \in [s]$, Γ_w is an r-loop of length 1, and $\pi := \{\{1\}\}$ is the only partition of the vertex set of Γ_w. It is clear that $\pi \in \mathrm{Con}(\Gamma_w)$, that $\chi(\Gamma_w/\pi) = 1$, and that $\pi \in \mathrm{SCon}(\Gamma_w)$ iff $d_r = 1$, and hence iff $w \approx e$.

Now suppose that the assertion holds for every word in $\mathbb{F} \setminus \{e\}$ having length less than or equal to some positive integer n, and let $w \in \mathbb{F} \setminus \{e\}$ have length $n + 1$.

First suppose that there exists $\pi \in \mathrm{SCon}(\Gamma_w)$ such that $\chi(\Gamma_w/\pi) = 1$. By Lemma 2.8, there exists a path P of Γ_w/π such that $(\Gamma_w/\pi) \setminus P$ has at most one non-trivial component. On the other hand, since $\chi(\Gamma_w/\pi) = 1$, it follows from Proposition 2.10 that P has exactly one vertex in common with each connected component of $(\Gamma_w/\pi) \setminus P$. Thus at most one vertex of P belongs to other paths of Γ_w/π, so by Lemma 2.16 some cyclic permutation of the symbols of w is of the form vx with x as in (2.3). To show that $w \approx e$, it suffices to show that $vx \approx e$. If $v = e$, this is immediate. If $v \neq e$, then there exists $\pi \in \mathrm{SCon}(\Gamma_{vx})$ such that $\chi(\Gamma_{vx}/\pi) = 1$ (since Γ_{vx} is isomorphic to Γ_w), so by Lemma 2.15(b) there exists $\pi \in \mathrm{SCon}(\Gamma_v)$ such that $\chi(\Gamma_v/\pi) = 1$; thus $v \approx e$ by the induction hypothesis, so $vx \approx e$.

Next suppose that $w \approx e$. Then some cyclic permutation of the symbols of w is of the form vx with $v \approx e$ and x as in (2.3). Since Γ_{vx} is isomorphic to Γ_w, it suffices to show that

$$\mathrm{card}\big\{\pi \in \mathrm{SCon}(\Gamma_{vx}) \,\big|\, \chi(\Gamma_{vx}/\pi) = 1\big\} = 1.$$

If $v = e$, this follows from Lemma 2.15(a), whereas if $v \neq e$, it follows from Lemma 2.15(b), since $\mathrm{card}\{\rho \in \mathrm{SCon}(\Gamma_v) \,|\, \chi(\Gamma_v/\rho) = 1\} = 1$ by the induction hypothesis. $\qquad\square$

3. Permutations with Restricted Cycle Lengths

3.1. Permutations with restricted cycle lengths. Let A be a non-empty set of positive integers. For every positive integer N, $\mathcal{S}_N$ denotes the set of all permutations of $[N]$ and $\mathcal{S}_N^{(A)}$ denotes the subset of $\mathcal{S}_N$ defined by

$$\mathcal{S}_N^{(A)} = \{\sigma \in \mathcal{S}_N \,|\, \text{the length of every cycle of } \sigma \text{ belongs to } A\}. \tag{3.1}$$

For every positive integer N, the cardinality of the set $\mathcal{S}_N^{(A)}$ is denoted by $a_N^{(A)}$; by convention, $a_0^{(A)} := 1$. It is well known (see, for instance, [1, Theorem 3.53]) that the exponential generating function for the numbers $a_N^{(A)}$ satisfies

$$\sum_{N=0}^{\infty} \frac{a_N^{(A)}}{N!} z^N = \exp\left(\sum_{k \in A} \frac{z^k}{k}\right). \tag{3.2}$$

It is clear that $a_N^{(A)} = 0$ for every N which is not a multiple of $\gcd(A)$, where $\gcd(A)$ denotes the greatest common divisor of the elements of A. On the other hand, we have the following.

3.2. Lemma. *If A is a non-empty set of positive integers, then $a_N^{(A)} > 0$ for every sufficiently large multiple N of $\gcd(A)$.*

Proof. Let $D = \gcd(A)$. Then there exist $k_1, k_2, \ldots, k_n \in A$ and integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$D = \sum_{i=1}^{n} \alpha_i k_i.$$

Let $N = \frac{k_1}{D} \sum_{i=1}^{N} |\alpha_i| \, k_i$. For every positive integer m, let q, r be non-negative integers such that $m = q\left(\frac{k_1}{D}\right) + r$ and $0 \le r < \frac{k_1}{D}$; then

$$N + mD = N + qk_1 + rD = qk_1 + \sum_{i=1}^{n} \left(\frac{k_1}{D} |\alpha_i| + r\alpha_i\right) k_i.$$

Thus every multiple of D which is greater than N can be written as a linear combination of elements of A with positive coefficients, from which the lemma follows. $\qquad\square$

3.3. Cycles of a given length. Let A be a non-empty set of positive integers, let $k \in A$, and let N be a positive integer such that $N \ge k$ and $\mathcal{S}_N^{(A)} \neq \emptyset$. For every $i, j \in [N]$, it is easily seen that the map

$$\left\{\sigma \in \mathcal{S}_N^{(A)} \,\middle|\, i \text{ belongs to a cycle of length } k\right\} \longrightarrow \left\{\sigma \in \mathcal{S}_N^{(A)} \,\middle|\, j \text{ belongs to a cycle of length } k\right\}$$
$$\sigma \longmapsto (i,j)\sigma(i,j)$$

is a bijection, so the quantity $p_N^{(A)}(k)$ defined by

$$p_N^{(A)}(k) = \frac{\text{card}\left\{\sigma \in \mathcal{S}_N^{(A)} \,\middle|\, i \text{ belongs to a cycle of length } k\right\}}{a_N^{(A)}} \tag{3.3}$$

(notation as in 3.1) is independent of the choice of $i \in [N]$. Clearly, $p_N^{(A)}(k)$ is the probability that, in a randomly chosen permutation from $\mathcal{S}_N^{(A)}$, i belongs to a cycle of length k; equivalently, $p_N^{(A)}(k)$ is $\frac{k}{N}$ times the expected number of cycles of length k of a randomly chosen permutation from $\mathcal{S}_N^{(A)}$. Note that for every $i \in [N]$, we have that

$$\operatorname{card}\{\sigma \in \mathcal{S}_N^{(A)} \,|\, i \text{ belongs to a cycle of length } k\} = (N-1)(N-2)\cdots(N-k+1)a_{N-k}^{(A)},$$
(3.4)

because there are $N - 1$ equally likely possibilities for $\sigma(i)$, and for each of these possibilities there are $N - 2$ equally likely possibilities for $\sigma^2(i)$ etc., and once the cycle (of length k) to which i belongs is determined, there are $a_{N-k}^{(A)}$ possibilities for the remaining cycles. Thus

$$p_N^{(A)}(k) = \frac{(N-1)!\, a_{N-k}^{(A)}}{(N-k)!\, a_N^{(A)}} = \frac{1}{N} \cdot \frac{a_{N-k}^{(A)}/(N-k)!}{a_N^{(A)}/N!}.$$
(3.5)

3.4. Remarks. Let A be a non-empty set of positive integers and let $k \in A$. We shall be interested in the asymptotic behavior of $p_N^{(A)}(k)$ as $N \to \infty$ with $\mathcal{S}_N^{(A)} \neq \emptyset$. This asymptotic behavior is described in Proposition 3.7 below, under the assumption that the set A satisfies the following condition:

$$\text{Either } A \text{ is finite, or } \sum_{j \in \mathbb{N} \setminus A} \frac{1}{j} < \infty.$$
(3.6)

[Note that the second hypothesis on A implies that $\gcd(A) = 1$, so $S_N^{(A)} \neq \emptyset$ for all sufficiently large N.] The part of Proposition 3.7 which deals with the case when A is finite relies on Lemma 3.6 below, which uses Hayman's method for determining the asymptotics of power series coefficients of certain complex analytic functions. In turn, Lemma 3.6 relies on the following intermediate step.

3.5. Lemma. *Let $\mathcal{N}$ be a neighborhood of the origin in the complex plane, let $F : \mathcal{N} \to \mathbb{C}$ be an analytic function with $F(0) = 1$, let k be a positive integer, let $C > 0$, and for every positive integer N with $t_N := \left(\frac{C}{N}\right)^{1/k} \in \mathcal{N}$, let $r_N = \frac{F(t_N)}{t_N}$. Then:*

(a) $\displaystyle\lim_{N\to\infty} \left(r_N^k - r_{N-1}^k\right) = \frac{1}{C}$.

(b) $\displaystyle\lim_{N\to\infty} \left(r_N^j - r_{N-1}^j\right) = 0$ for every $0 \leq j < k$.

(c) $\dfrac{r_N^N}{r_{N-1}^{N-1}} \sim \left(\dfrac{eN}{C}\right)^{1/k}$ as $N \to \infty$.

Proof. The proof uses the simple observation that

$$\lim_{N\to\infty} \left(N^\alpha - (N-1)^\alpha\right) = 0 \qquad \text{for every } 0 \leq \alpha < 1,$$

which immediately implies that

$$\lim_{N\to\infty} \left(\frac{1}{t_N^m} - \frac{1}{t_{N-1}^m}\right) = 0 \qquad \text{for every } 0 \leq m < k.$$
(3.7)

To prove (a) and (b), let $0 \leq j \leq k$ and consider the power series expansion of F^j around the origin:

$$F(t)^j =: 1 + \sum_{i=1}^{j} \alpha_i t^i + t^{j+1} G_j(t).$$

Then for every sufficiently large N, we have that

$$r_N^j = \frac{F(t_N)^j}{t_N^j} = \frac{1}{t_N^j} + \sum_{i=1}^{j} \frac{\alpha_i}{t_N^{j-i}} + t_N G_j(t_N),$$

so (3.7) implies that

$$\lim_{N \to \infty} \left(r_N^j - r_{N-1}^j \right) = \lim_{N \to \infty} \left(\frac{1}{t_N^j} - \frac{1}{t_{N-1}^j} \right) = \begin{cases} \frac{1}{C} & \text{if } j = k \\ 0 & \text{if } j < k. \end{cases}$$

To prove (c), consider the power series expansion of $\log F$ around the origin:

$$\log F(t) =: \sum_{i=1}^{k} \beta_i t^i + t^{k+1} G(t).$$

Then for every sufficiently large N, we have that

$$F(t_N)^N = \exp\left(N \log F(t_N)\right) = \exp\left(\frac{C}{t_N^k} \log F(t_N)\right) = \exp\left(\sum_{i=1}^{k} \frac{C\beta_i}{t_N^{k-i}} + Ct_N G(t_N)\right),$$

so (3.7) implies that

$$\lim_{N \to \infty} \frac{F(t_N)^N}{F(t_{N-1})^{N-1}} = 1,$$

from which it follows that

$$\frac{r_N^N}{r_{N-1}^{N-1}} \sim \frac{t_{N-1}^{N-1}}{t_N^N} = \left(\frac{N^N}{C(N-1)^{N-1}}\right)^{1/k} \sim \left(\frac{eN}{C}\right)^{1/k} \qquad \text{as } N \to \infty.$$

$\square$

3.6. Lemma. *Let n be a positive integer, let $c_1, c_2, \ldots, c_n > 0$, let $(k_i)_{i=1}^{n}$ be an increasing sequence of positive integers, let $\sum_{N=0}^{\infty} b_N w^N$ be the power series expansion around the origin of the function $f(w) := \exp\left(\sum_{i=1}^{n} c_i w^{k_i}\right)$, and suppose that $b_N > 0$ for all sufficiently large N. Then*

$$\frac{b_{N-1}}{b_N} \sim \left(\frac{N}{c_n k_n}\right)^{1/k_n} \qquad \text{as } N \to \infty. \tag{3.8}$$

Proof. The proof relies on Hayman's method for determining the asymptotics of power series coefficients of certain complex analytic functions, which is described in detail in [9, Section 5.4]. By Theorem 5.4.1 of that work, we have that

$$b_N \sim \frac{f(r_N)}{r_N^N \sqrt{2\pi \left(\sum\limits_{i=1}^n c_i k_i^2 r_N^{k_i} \right)}} \qquad \text{as } N \to \infty, \tag{3.9}$$

where r_N is the positive real root of the equation

$$\sum_{i=1}^n c_i k_i r_N^{k_i} = N. \tag{3.10}$$

It is immediate that $\lim\limits_{N\to\infty} r_N = \infty$, and hence that

$$\sum_{i=1}^n c_i k_i^2 r_N^{k_i} \sim c_n k_n^2 r_N^{k_n} \sim k_n \left(\sum_{i=1}^n c_i k_i r_N^{k_i} \right) = k_n N \qquad \text{as } N \to \infty,$$

so (3.9) implies that

$$\frac{b_{N-1}}{b_N} \sim \frac{f(r_{N-1})}{f(r_N)} \cdot \frac{r_N^N}{r_{N-1}^{N-1}} = \frac{r_N^N}{r_{N-1}^{N-1}} \exp\left(-\sum_{i=1}^n c_i \left(r_N^{k_i} - r_{N-1}^{k_i} \right) \right) \qquad \text{as } N \to \infty. \tag{3.11}$$

To evaluate the right-hand side of (3.11), further analysis of (3.10) is needed. First note that (3.10) can be put in the form

$$\frac{1}{r_N} = \left(\frac{c_n k_n}{N} \right)^{1/k_n} \left(1 + \sum_{i=1}^{n-1} \frac{c_i k_i}{c_n k_n r_N^{k_n - k_i}} \right)^{1/k_n}.$$

Via the generalized binomial expansion of the right-hand side and with the substitutions $u_N := \frac{1}{r_N}$ and $t_N := \left(\frac{c_n k_n}{N} \right)^{1/k_n}$, this can in turn be written as

$$u_N = t_N \left(1 + \sum_{j=1}^\infty \alpha_j u_N^j \right)$$

for some coefficients α_j such that the series $\varphi(u) := 1 + \sum\limits_{j=1}^\infty \alpha_j u^j$ has positive radius of convergence. Since the function $\frac{u}{\varphi(u)}$ is conformal for sufficiently small u, it has an analytic inverse g [with $g(0) = 0$ and $g'(0) = 1$] in a small neighborhood of the origin. Thus $u_N = g(t_N)$ for all sufficiently large N. Letting $F(t) := \frac{t}{g(t)}$ for all sufficiently small t and $F(0) := 1$, it follows that F is analytic and $r_N = \frac{F(t_N)}{t_N}$ for all large enough N, so (3.11) and Lemma 3.5 imply that

$$\frac{b_{N-1}}{b_N} \sim \left(\frac{eN}{c_n k_n} \right)^{1/k_n} \exp(-c_n/c_n k_n) = \left(\frac{N}{c_n k_n} \right)^{1/k_n} \qquad \text{as } N \to \infty.$$

$\square$

3.7. Proposition. *Let A be a non-empty set of positive integers which satisfies condition (3.6), let $k \in A$, let $d = \sup A$, and for every positive integer N with $S_N^{(A)} \neq \emptyset$, let $p_N^{(A)}(k)$ be as in 3.3. Then*

$$p_N^{(A)}(k) \sim N^{(k/d)-1} \qquad \text{as } N \to \infty \text{ with } S_N^{(A)} \neq \emptyset \tag{3.12}$$

(with the convention that $k/\infty := 0$).

Proof. Suppose first that A is finite and that the elements of A are $k_1, k_2, \dots, k_n$, written in increasing order (so $d = k_n$). Let $D = \gcd(A)$. Then

$$\sum_{N=0}^{\infty} \frac{a_{DN}^{(A)}}{(DN)!} z^{DN} = \exp\left(\sum_{i=1}^{n} \frac{z^{k_i}}{k_i}\right), \qquad \text{[by (3.2)]}$$

which, via the substitution $w := z^D$, can be rewritten as

$$\sum_{N=0}^{\infty} \frac{a_{DN}^{(A)}}{(DN)!} w^N = \exp\left(\sum_{i=1}^{n} \frac{w^{k_i/D}}{k_i}\right).$$

Since $a_{DN}^{(A)} > 0$ for all sufficiently large N by Lemma 3.2, it follows from Lemma 3.6 that

$$\frac{a_{DN-D}^{(A)}/(DN-D)!}{a_{DN}^{(A)}/(DN)!} \sim (DN)^{D/k_n} \qquad \text{as } N \to \infty.$$

Hence (3.5) implies that

$$p_{DN}^{(A)}(k) = \frac{1}{DN} \cdot \frac{a_{DN-k}^{(A)}/(DN-k)!}{a_{DN}^{(A)}/(DN)!} \sim (DN)^{(k/k_n)-1} \qquad \text{as } N \to \infty.$$

Next suppose that $\sum_{j \in \mathbb{N} \setminus A} \frac{1}{j} < \infty$ [which implies that $d = \infty$ and that $\gcd(A) = 1$, so $S_N^{(A)} \neq \emptyset$ for all sufficiently large N]. For every $N \geq 1$ we have that

$$a_N^{(A)} = \sum_{j \in A \cap [N]} \text{card}\{\sigma \in \mathcal{S}_N^{(A)} \mid 1 \text{ belongs to a cycle of length } j\} = \sum_{j \in A \cap [N]} \frac{(N-1)!\, a_{N-j}^{(A)}}{(N-j)!}$$

$$\text{[by (3.4)]},$$

so

$$\frac{a_N^{(A)}}{N!} = \frac{1}{N} \sum_{j \in A \cap [N]} \frac{a_{N-j}^{(A)}}{(N-j)!}.$$

Because of the assumption $\sum_{j \in \mathbb{N} \setminus A} \frac{1}{j} < \infty$, it now follows immediately from Theorem 1 of [2] that

$$\lim_{N \to \infty} \frac{a_N^{(A)}}{N!} = \exp\left(-\sum_{j \in \mathbb{N} \setminus A} \frac{1}{j}\right).$$

(This conclusion also follows from Theorem 1.1 of [3], in the more general context of combinatorial assemblies.) Hence (3.5) implies that

$$p_N^{(A)}(k) = \frac{1}{N} \cdot \frac{a_{N-k}^{(A)}/(N-k)!}{a_N^{(A)}/N!} \sim \frac{1}{N} \qquad \text{as } N \to \infty.$$

$\square$

3.8. Remarks. (a) Condition (3.6) is not necessary for the conclusion of Proposition 3.7 to hold, as shown by the following example.

Let D be a positive integer and let $A = \{DN \mid N \geq 1\}$. By (3.2), we have that

$$\sum_{N=0}^{\infty} \frac{a_{DN}^{(A)}}{(DN)!} z^{DN} = \exp\left(\sum_{N=1}^{\infty} \frac{z^{DN}}{DN}\right) = \exp\left(-\frac{1}{D}\log\left(1-z^D\right)\right) = \left(1-z^D\right)^{-1/D}$$

$$= \sum_{N=0}^{\infty} \binom{-1/D}{N} \left(-z^D\right)^N,$$

where $\binom{-1/D}{N} := \frac{1}{N!}\left(-\frac{1}{D}\right)\left(-\frac{1}{D}-1\right)\cdots\left(-\frac{1}{D}-N+1\right)$ is a generalized binomial coefficient. Thus

$$\frac{a_{DN}^{(A)}}{(DN)!} = \frac{1}{N!}\left(\frac{1}{D}\right)\left(\frac{1}{D}+1\right)\cdots\left(\frac{1}{D}+N-1\right) \qquad \forall N \geq 1,$$

from which it immediately follows that

$$\lim_{N\to\infty} \frac{a_{DN-D}^{(A)}/(DN-D)!}{a_{DN}^{(A)}/(DN)!} = 1.$$

Hence (3.5) implies that for every $k \in A$, we have that

$$p_{DN}^{(A)}(k) = \frac{1}{DN} \cdot \frac{a_{DN-k}^{(A)}/(DN-k)!}{a_{DN}^{(A)}/(DN)!} \sim \frac{1}{DN} \qquad \text{as } N \to \infty.$$

(b) On the other hand, the following example shows the existence of sets of positive integers for which the conclusion of Proposition 3.7 does not hold.

First let us construct inductively a sequence $(N_k)_{k=1}^{\infty}$ of positive integers such that

$$N_{k+1} p_{N_{k+1}}^{(A_k)}(1) > 2 \qquad \forall k \geq 1,$$

where $A_k := \{1, N_1+1, N_2+1, \ldots, N_k+1\}$. Let N_1 be any positive integer. If $k \geq 1$ and $N_1, N_2, \ldots, N_k$ have been chosen, Proposition 3.7 implies that $p_N^{(A_k)}(1) \sim N^{1/(N_k+1)-1}$ as $N \to \infty$. In particular, $Np_N^{(A_k)}(1) \to \infty$ as $N \to \infty$, so there exists $N_{k+1} > N_k$ such that $N_{k+1} p_{N_{k+1}}^{(A_k)} > 2$.

Now let $A = \bigcup_{k=1}^{\infty} A_k$. Since $p_N^{(A)}(1)$ depends only on the elements of A which are less than or equal to N, we have that $p_{N_{k+1}}^{(A)}(1) = p_{N_{k+1}}^{(A_k)}(1) \ \forall k \geq 1$. Thus $N_{k+1} p_{N_{k+1}}^{(A)}(1) > 2 \ \forall k \geq 1$, so $p_N^{(A)}(1) \not\sim \frac{1}{N}$ as $N \to \infty$.

3.9. Permutations compatible with graphs. (a) For every non-empty set A of positive integers, an A-*graph* is a monocolored admissible graph Γ with the properties that $l(L) \in A$ for every loop L of Γ and $l(S) < \sup A$ for every string S of Γ (notation and terminology as in 2.7 and 2.9).

(b) Let A be a non-empty set of positive integers, let N be a positive integer such that $\mathcal{S}_N^{(A)} \neq \emptyset$, and let Γ be a monocolored graph with vertex set contained in $[N]$. A permutation $\sigma \in \mathcal{S}_N^{(A)}$ is said to be *compatible with* Γ iff for every edge $i \to j$ of Γ, we have that $\sigma(i) = j$. The set of all permutations in $\mathcal{S}_N^{(A)}$ which are compatible with Γ is denoted by $\mathcal{S}_N^{(A)}(\Gamma)$. It is clear that, if Γ is not an A-graph, then $\mathcal{S}_N^{(A)}(\Gamma) = \emptyset$.

(c) Let A be a non-empty set of positive integers, let N be a positive integer such that $\mathcal{S}_N^{(A)} \neq \emptyset$, and let Γ be an A-graph with vertex set V consisting of at most N vertices $v_1, v_2, \dots, v_k$. For every injective function $f : V \to [N]$, let $f(\Gamma)$ denote the graph obtained from Γ by relabeling $v_1, v_2, \dots, v_k$ as $f(v_1), f(v_2), \dots, f(v_k)$, respectively. If f and g are two such functions, it is easily seen that the map

$$\mathcal{S}_N^{(A)}(f(\Gamma)) \longrightarrow \mathcal{S}_N^{(A)}(g(\Gamma)),$$

$$\sigma \longmapsto \big(f(v_1), g(v_1)\big) \cdots \big(f(v_k), g(v_k)\big) \sigma \big(f(v_1), g(v_1)\big) \cdots \big(f(v_k), g(v_k)\big)$$

is a bijection, so the quantity $p_N^{(A)}(\Gamma)$ defined by

$$p_N^{(A)}(\Gamma) = \frac{\operatorname{card} \mathcal{S}_N^{(A)}(f(\Gamma))}{a_N^{(A)}} \tag{3.13}$$

is independent of the choice of the injective function $f : V \to [N]$. Clearly, $p_N^{(A)}(\Gamma)$ is the probability that a randomly chosen permutation from $\mathcal{S}_N^{(A)}$ is compatible with $f(\Gamma)$.

3.10. Proposition. *Let A be a non-empty set of positive integers which satisfies condition (3.6), let $d = \sup A$, let Γ be an A-graph [terminology as in 3.9(a)], and for every positive integer N with $S_N^{(A)} \neq \emptyset$, let $p_N^{(A)}(\Gamma)$ be as in 3.9(c). Then*

$$p_N^{(A)}(\Gamma) \sim N^{-E(\Gamma)+L(\Gamma)+ \sum\limits_{\substack{loops \\ P \ of \ \Gamma}} \left[\frac{l(P)}{d}-1\right]} \qquad \text{as } N \to \infty \text{ with } \mathcal{S}_N^{(A)} \neq \emptyset \tag{3.14}$$

$\left(\text{with the convention that } \frac{l(P)}{\infty} := 0\right).$

Proof. The proof is by induction on the number of paths of Γ. If Γ has no paths (and therefore no edges), then it easily seen that $p_N^{(A)}(\Gamma) = 1$ for every positive integer N with $\mathcal{S}_N^{(A)} \neq \emptyset$.

Now suppose that the assertion of the proposition holds for every A-graph with number of paths less than some positive integer n, and let Γ be an A-graph with n paths. Without loss of generality, we may assume that the vertex set of Γ is contained in $[N_0]$ for some positive integer N_0. Let Q be any path of Γ and let $\Gamma \setminus Q$ be the graph obtained from Γ by removing all edges of Q.

First suppose that Q is a loop $v_1 \to v_2 \to \cdots \to v_k \to v_1$ (of some length $k \in A$). Then for every $N \geq N_0$ with $\mathcal{S}_N^{(A)} \neq \emptyset$, we have that

$$p_N^{(A)}(\Gamma) = p_N^{(A)}(Q) \cdot p_{N-k}^{(A)}(\Gamma \setminus Q)$$

and that

$$p_N^{(A)}(Q) = \frac{p_N^{(A)}(k)}{(N-1)(N-2)\cdots(N-k+1)},$$

because, given that v_1 belongs to a cycle of length k of a permutation $\sigma \in \mathcal{S}_N^{(A)}$, there are $N-1$ equally likely possibilities for $\sigma(v_1)$, and for each of these possibilities there are $N-2$ equally likely possibilities for $\sigma^2(v_1)$ etc. Then Proposition 3.7 and the induction hypothesis imply that

$$p_N^{(A)}(\Gamma) \sim N^{-k+1} N^{(k/d)-1} N^{-E(\Gamma \setminus Q)+L(\Gamma \setminus Q)+\sum_{\substack{\text{loops } P \\ \text{of } \Gamma \setminus Q}} \left[\frac{l(P)}{d}-1\right]}$$

$$= N^{-E(\Gamma)+L(\Gamma)+\sum_{\substack{\text{loops} \\ P \text{ of } \Gamma}} \left[\frac{l(P)}{d}-1\right]} \qquad \text{as } N \to \infty \text{ with } S_N^{(A)} \neq \emptyset.$$

Finally, suppose that Q is a string $v_1 \to v_2 \to \cdots \to v_k \to v_{k+1}$ (of some length $k < \sup A$). Let V be the set of vertices which belong to the paths of Γ other than Q, and let $|V|$ be the number of vertices in V. Let q_N be the probability that a randomly chosen permutation $\sigma \in \mathcal{S}_N^{(A)}(\Gamma \setminus Q)$ has the property that the numbers $\sigma(v_1), \sigma^2(v_1), \ldots, \sigma^k(v_1)$ are distinct from each other and from the vertices in V. Then for every $N \geq N_0$ with $\mathcal{S}_N^{(A)} \neq \emptyset$, we have that

$$p_N^{(A)}(\Gamma) = \frac{p_N^{(A)}(\Gamma \setminus Q) \cdot q_N}{(N-|V|-1)(N-|V|-2)\cdots(N-|V|-k)},$$

because there are $N-|V|-1$ equally likely possibilities for $\sigma(v_1)$, and for each of these possibilities there are $N-|V|-2$ equally likely possibilities for $\sigma^2(v_1)$ etc. Then the induction hypothesis implies that

$$p_N^{(A)}(\Gamma) \sim q_N \cdot N^{-k} N^{-E(\Gamma \setminus Q)+L(\Gamma \setminus Q)+\sum_{\substack{\text{loops } P \\ \text{of } \Gamma \setminus Q}} \left[\frac{l(P)}{d}-1\right]}$$

$$= q_N \cdot N^{-E(\Gamma)+L(\Gamma)+\sum_{\substack{\text{loops} \\ P \text{ of } \Gamma}} \left[\frac{l(P)}{d}-1\right]} \qquad \text{as } N \to \infty \text{ with } S_N^{(A)} \neq \emptyset,$$

so it suffices to show that $q_N \to 1$ (or equivalently, that $1 - q_N \to 0$) as $N \to \infty$ with $\mathcal{S}_N^{(A)} \neq \emptyset$.

Let $N \geq N_0$ be such that $\mathcal{S}_N^{(A)} \neq \emptyset$. Given a permutation $\sigma \in \mathcal{S}_N^{(A)}(\Gamma \setminus Q)$, there are only two (mutually exclusive) ways in which σ can fail to have the property that the numbers $\sigma(v_1), \sigma^2(v_1), \ldots, \sigma^k(v_1)$ are distinct from each other and from the vertices in V; namely, either

(1) at least one of $\sigma(v_1), \sigma^2(v_1), \dots, \sigma^k(v_1)$ belongs to V, or

(2) v_1 belongs to a cycle of σ of length at most k which does not contain any vertex in V.

For any vertex $v \in V$ and for every $i \in [k]$, the number of possibilities for $\sigma^{-i}(v)$ is either 1 or $N - |V|$ (and these possibilities are equally likely), so the probability $q_N^{(1)}$ of (1) occurring satisfies

$$q_N^{(1)} \leq \sum_{v \in V} \sum_{i=1}^{k} \frac{1}{N - |V|} = \frac{k \cdot |V|}{N - |V|}.$$

On the other hand, the probability $q_N^{(2)}$ of (2) occurring satisfies

$$q_N^{(2)} \leq \sum_{j \in A \cap [k]} \frac{p_N^{(A)}(j) \cdot p_{N-j}^{(A)}(\Gamma \setminus Q)}{p_N^{(A)}(\Gamma \setminus Q)},$$

which, by the induction hypothesis and Proposition 3.7 (together with the fact that $k < \sup A$), converges to 0 as $N \to \infty$ with $\mathcal{S}_N^{(A)} \neq \emptyset$.

Thus $1 - q_N = q_N^{(1)} + q_N^{(2)} \to 0$ as $N \to \infty$ with $\mathcal{S}_N^{(A)} \neq \emptyset$. $\square$

4. ASYMPTOTIC FREENESS.

4.1. Non-commutative $*$-probability spaces. (a) A *non-commutative $*$-probability space* (or, simply, a *$*$-probability space*) is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital $*$-algebra (the unit of which is denoted by 1) and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional which satisfies

$$\varphi(1) = 1 \qquad \text{and} \qquad \varphi(a^*a) \geq 0 \quad \forall a \in \mathcal{A}. \tag{4.1}$$

Elements of $\mathcal{A}$ are called *non-commutative random variables* (or, simply, *random variables*).

(b) Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. A family $(\mathcal{A}_j)_{j \in J}$ of unital subalgebras of $\mathcal{A}$ is said to be *free* iff for every integer $n \geq 1$, for every $j_1, j_2, \dots, j_n \in J$ such that $j_1 \neq j_2 \neq \cdots \neq j_n$, and for every $a_1 \in \mathcal{A}_{j_1}$, $a_2 \in \mathcal{A}_{j_2}$, $\dots, a_n \in \mathcal{A}_{j_n}$ such that $\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_n) = 0$, we have that $\varphi(a_1 a_2 \cdots a_n) = 0$. More generally, a family $(a_j)_{j \in J}$ of random variables in $\mathcal{A}$ is said to be *free* (respectively, *$*$-free*) iff the family $(\mathcal{A}_j)_{j \in J}$ is free, where each $\mathcal{A}_j$ is the unital subalgebra (respectively, $*$-subalgebra) of $\mathcal{A}$ generated by a_j.

(c) Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. If d is a positive integer, a *d-Haar unitary* in $(\mathcal{A}, \varphi)$ is a unitary random variable $u \in \mathcal{A}$ such that $u^d = 1$ and $\varphi(u^n) = 0$ whenever n is not a multiple of d. An *∞-Haar unitary* in $(\mathcal{A}, \varphi)$ is a unitary random variable $u \in \mathcal{A}$ such that $\varphi(u^n) = 0 \ \forall n \neq 0$.

4.2. Random matrices and convergence in $*$-distribution. (a) For every positive integer N, the set of all $N \times N$ matrices with complex entries is denoted by $\mathcal{M}_N$. If $A \in \mathcal{M}_N$, then tr A denotes the *normalized trace* of A:

$$\operatorname{tr} A := \frac{1}{N} \sum_{i=1}^{N} A_{ii}. \tag{4.2}$$

(b) We shall work with a fixed probability space $(\Omega, \mathcal{F}, P)$, over which we shall consider *random variables* and *random $N \times N$ matrices*; that is, measurable functions $f : \Omega \to \mathbb{C}$ and $A : \Omega \to \mathcal{M}_N$. The *expectation* of an integrable random variable f over $(\Omega, \mathcal{F}, P)$ is

$$\mathbb{E}(f) := \int_\Omega f(\omega) \, \mathrm{d}P(\omega). \tag{4.3}$$

(c) Let $(a_r)_{r=1}^s$ be a family of random variables in a $*$-probability space $(\mathcal{A}, \varphi)$, let $(N_k)_{k=1}^\infty$ be an increasing sequence of positive integers, and for each k let $\left(A_r^{(k)}\right)_{r=1}^s$ be a family of random $N_k \times N_k$ matrices over $(\Omega, \mathcal{F}, P)$. The families $\left(A_r^{(k)}\right)_{r=1}^s$ are said to *converge in $*$-distribution* to $(a_r)_{r=1}^s$ as $k \to \infty$, written $\left(A_r^{(k)}\right)_{r=1}^s \xrightarrow[k\to\infty]{*\text{-distr.}} (a_r)_{r=1}^s$, iff

$$\lim_{k\to\infty} \mathbb{E}\left[\mathrm{tr}\,\left(A_{r_1}^{(k)}\right)^{\varepsilon_1}\left(A_{r_2}^{(k)}\right)^{\varepsilon_2}\cdots\left(A_{r_n}^{(k)}\right)^{\varepsilon_n}\right] = \varphi\left(a_{r_1}^{\varepsilon_1}a_{r_2}^{\varepsilon_2}\cdots a_{r_n}^{\varepsilon_n}\right) \tag{4.4}$$

for every positive integer n, for every $r_1, r_2, \ldots, r_n \in [s]$, and for every $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{1, *\}$.

The families $\left(A_r^{(k)}\right)_{r=1}^s$ are said to *converge in $*$-distribution almost surely* to $(a_r)_{r=1}^s$ as $k \to \infty$, written $\left(A_r^{(k)}\right)_{r=1}^s \xrightarrow[k\to\infty]{*\text{-distr. a.s.}} (a_r)_{r=1}^s$, iff they satisfy the stronger condition that

$$\lim_{k\to\infty} \mathrm{tr}\,\left(A_{r_1}^{(k)}\right)^{\varepsilon_1}\left(A_{r_2}^{(k)}\right)^{\varepsilon_2}\cdots\left(A_{r_n}^{(k)}\right)^{\varepsilon_n} = \varphi\left(a_{r_1}^{\varepsilon_1}a_{r_2}^{\varepsilon_2}\cdots a_{r_n}^{\varepsilon_n}\right) \qquad \text{almost surely} \tag{4.5}$$

for every positive integer n, for every $r_1, r_2, \ldots, r_n \in [s]$, and for every $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{1, *\}$.

In many concrete instances where convergence in $*$-distribution is known to hold, it can be shown that it holds almost surely by means of the following fact, the proof of which can be found, for instance, embedded in the proof of Corollary 3.9 in [7].

4.3. Proposition. *Let $\lambda \in \mathbb{C}$ and let $(f_k)_{k=1}^\infty$ be a sequence of random variables over $(\Omega, \mathcal{F}, P)$ such that $\lim\limits_{k\to\infty} \mathbb{E}(f_k) = \lambda$ and such that*

$$\sum_{k=1}^\infty \left[\mathbb{E}\left(|f_k|^2\right) - \left|\mathbb{E}(f_k)\right|^2\right] < \infty. \tag{4.6}$$

Then $\lim\limits_{k\to\infty} f_k = \lambda$ almost surely.

4.4. Random permutation matrices. (a) For every positive integer N and for every $\sigma \in \mathcal{S}_N$, M_σ denotes the $N \times N$ matrix defined by

$$(M_\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{if } \sigma(j) \neq i \end{cases} \qquad \forall i, j \in [N]. \tag{4.7}$$

(b) Let N be a positive integer. A *random $N \times N$ permutation matrix* over $(\Omega, \mathcal{F}, P)$ is a random $N \times N$ matrix over $(\Omega, \mathcal{F}, P)$ with range contained in the set $\{M_\sigma \,|\, \sigma \in \mathcal{S}_N\}$.

If $\mathcal{S}$ is a non-empty subset of $\mathcal{S}_N$, a random $N \times N$ permutation matrix U over $(\Omega, \mathcal{F}, P)$ is said to be *uniformly distributed over $\mathcal{S}$* iff its range is the set $\{M_\sigma \,|\, \sigma \in \mathcal{S}\}$ and

$$P\left[U^{-1}(M_\sigma)\right] = \frac{1}{\mathrm{card}\,\mathcal{S}} \qquad \forall \sigma \in \mathcal{S}. \tag{4.8}$$

4.5. Notation. Throughout the remainder of this section, the following are fixed:

(a) A $*$-free family $(u_r)_{r=1}^s$ of random variables in a $*$-probability space $(\mathcal{A}, \varphi)$ such that each u_r is a d_r-Haar unitary [notation as in 2.1(a)]. For every $w \in \mathbb{F} \setminus \{e\}$ [notation as in 2.1(b)], u_w denotes the random variable in $(\mathcal{A}, \varphi)$ obtained from w by replacing each g_r by u_r and each g_r^* by u_r^*; by convention, $u_e := 1$ (the unit of the algebra $\mathcal{A}$).

It is immediate from the definitions that for every $w \in \mathbb{F}$, we have that

$$\varphi(u_w) = \begin{cases} 1 & \text{if } w \approx e \\ 0 & \text{if } w \not\approx e. \end{cases} \tag{4.9}$$

(b) A sequence $A_1, A_2, \ldots, A_s$ of (not necessarily disjoint, or even distinct) non-empty sets of positive integers such that $\sup A_r = d_r \ \forall r \in [s]$ and such that each A_r satisfies condition (3.6), and an increasing sequence $(N_k)_{k=1}^\infty$ such that $\mathcal{S}_{N_k}^{(A_r)} \neq \emptyset \ \forall k \geq 1, \forall r \in [s]$ (notation as in 3.1). For every graph Γ, $\mathcal{C}(\Gamma)$ denotes the set of all congruences $\pi \in \mathrm{Con}(\Gamma)$ such that $(\Gamma/\pi)(r)$ is an A_r-graph for every $r \in [s]$ [notation and terminology as in 2.3(a), 2.5, 2.7(b), and 3.9(a)].

(c) For every positive integer k, an independent family $\left(U_r^{(k)}\right)_{r=1}^s$ of random $N_k \times N_k$ permutation matrices over $(\Omega, \mathcal{F}, P)$ such that each $U_r^{(k)}$ is uniformly distributed over $\mathcal{S}_{N_k}^{(A_r)}$. The independence requirement implies that for every positive integer k and for every measurable function $g : (\mathcal{M}_N)^s \to \mathbb{C}$, we have [with the notation from 3.1 and 4.4(a)] that

$$\int_\Omega g\left(U_1^{(k)}(\omega), U_2^{(k)}(\omega), \ldots, U_s^{(k)}(\omega)\right) \, dP(\omega) = \frac{1}{\prod\limits_{r=1}^s a_{N_k}^{(A_r)}} \sum_{\substack{\sigma_1 \in \mathcal{S}_{N_k}^{(A_1)} \\ \sigma_2 \in \mathcal{S}_{N_k}^{(A_2)} \\ \vdots \\ \sigma_s \in \mathcal{S}_{N_k}^{(A_s)}}} g(M_{\sigma_1}, M_{\sigma_2}, \ldots, M_{\sigma_s}).$$

$$\tag{4.10}$$

For every $w \in \mathbb{F} \setminus \{e\}$ [notation as in 2.1(b)] and for every positive integer k, $U_w^{(k)}$ denotes the random $N_k \times N_k$ (permutation) matrix over $(\Omega, \mathcal{F}, P)$ obtained from w by replacing each g_r by $U_r^{(k)}$ and each g_r^* by $\left(U_r^{(k)}\right)^*$; by convention, $U_e^{(k)}$ is the constant random matrix equal to $I^{(k)}$ (the $N_k \times N_k$ identity matrix).

If each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite, then it is easily seen that for every positive integer k and for every $w \in \mathbb{F}$ such that $w \approx e$, we have that $U_w^{(k)}$ is the constant random matrix equal to the $I^{(k)}$.

The main result of this paper is the following.

4.6. Theorem. *With the notation from 4.5 and 4.2, we have that:*

(a) $\left(U_r^{(k)}\right)_{r=1}^s \xrightarrow[k \to \infty]{*\text{-}distr.} (u_r)_{r=1}^s.$

(b) If each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite, then $\left(U_r^{(k)}\right)_{r=1}^s \xrightarrow[k \to \infty]{*\text{-}distr. \ a.s.} (u_r)_{r=1}^s.$

4.7. Remarks. Theorem 4.6 is equivalent to the statement that

$$\lim_{k\to\infty} \mathbb{E}\left(\operatorname{tr} U_w^{(k)}\right) = \varphi(u_w) \qquad \forall w \in \mathbb{F} \setminus \{e\} \tag{4.11}$$

and that, if each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite, then

$$\lim_{k\to\infty} \operatorname{tr} U_w^{(k)} = \varphi(u_w) \quad \text{almost surely} \qquad \forall w \in \mathbb{F} \setminus \{e\}. \tag{4.12}$$

By Proposition 4.3, (4.12) follows from (4.11) provided that it is shown that, if each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite, then

$$\sum_{k=1}^{\infty} \left[\mathbb{E}\left(\left|\operatorname{tr} U_w^{(k)}\right|^2\right) - \left|\mathbb{E}\left(\operatorname{tr} U_w^{(k)}\right)\right|^2\right] < \infty \qquad \forall w \in \mathbb{F} \setminus \{e\}. \tag{4.13}$$

In turn, (4.13) is a particular case of the more general statement that, if each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite, then

$$\sum_{k=1}^{\infty} \left[\mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)} \cdot \operatorname{tr} U_{w_2}^{(k)}\right) - \mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)}\right) \cdot \mathbb{E}\left(\operatorname{tr} U_{w_2}^{(k)}\right)\right] < \infty \qquad \forall w_1, w_2 \in \mathbb{F} \setminus \{e\}. \tag{4.14}$$

The following lemma gives an explicit formula for the type of expectations appearing in the above equations.

4.8. Lemma. *With the notation from 4.5, 2.3(a), 2.4, 2.5, and 3.9, let n and k be positive integers and let $w_1, w_2, \ldots, w_n \in \mathbb{F} \setminus \{e\}$. Then*

$$\mathbb{E}\left(\prod_{a=1}^{n} \operatorname{tr} U_{w_a}^{(k)}\right) = \frac{1}{N_k^n} \sum_{\pi \in \mathcal{C}(\Gamma)} \frac{N_k!}{(N_k - |\pi|)!} \prod_{r=1}^{s} p_{N_k}^{(A_r)}[(\Gamma/\pi)(r)], \tag{4.15}$$

where Γ denotes some (arbitrary) graph having n distinct connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ with $\Gamma_a \cong \Gamma_{w_a} \; \forall a \in [n]$. [The actual choice of Γ is irrelevant, since the right-hand side of (4.15) depends only on the isomorphism class of Γ.]

Proof. Write $w_1 = u_{r_1}^{\varepsilon_1} u_{r_2}^{\varepsilon_2} \cdots u_{r_{l_1}}^{\varepsilon_{l_1}}, w_2 = u_{r_{l_1+1}}^{\varepsilon_{l_1+1}} u_{r_{l_1+2}}^{\varepsilon_{l_1+2}} \cdots u_{r_{l_2}}^{\varepsilon_{l_2}}, \ldots, w_n = u_{r_{l_{n-1}+1}}^{\varepsilon_{l_{n-1}+1}} u_{r_{l_{n-1}+2}}^{\varepsilon_{l_{n-1}+2}} \cdots u_{r_{l_n}}^{\varepsilon_{l_n}}$, where $l_1 < l_2 < \cdots < l_n$, $r_1, r_2, \ldots, r_{l_n} \in [s]$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{l_n} \in \{1, *\}$. Then for every positive integer k, we have (with the convention that $l_0 := 0$) that

$$\mathbb{E}\left(\prod_{a=1}^{n} \operatorname{tr} U_{w_a}^{(k)}\right) = \mathbb{E}\left[\prod_{a=1}^{n} \operatorname{tr} \left(U_{r_{l_{a-1}+1}}^{(k)}\right)^{\varepsilon_{l_{a-1}+1}} \left(U_{r_{l_{a-1}+2}}^{(k)}\right)^{\varepsilon_{l_{a-1}+2}} \cdots \left(U_{r_{l_a}}^{(k)}\right)^{\varepsilon_{l_a}}\right]$$

$$= \int_{\Omega} \frac{1}{N_k^n} \sum_{i:[l_n]\to[N_k]} \prod_{a=1}^{n} \left[\left(U_{r_{l_{a-1}+1}}^{(k)}(\omega)^{\varepsilon_{l_{a-1}+1}}\right)_{i(l_{a-1}+1), i(l_{a-1}+2)} \cdot\right.$$

$$\left. \cdot \left(U_{r_{l_{a-1}+2}}^{(k)}(\omega)^{\varepsilon_{l_{a-1}+2}}\right)_{i(l_{a-1}+2), i(l_{a-1}+3)} \cdots \left(U_{r_{l_a}}^{(k)}(\omega)^{\varepsilon_{l_a}}\right)_{i(l_a), i(l_{a-1}+1)}\right] \mathrm{d}P(\omega)$$

$$= \frac{1}{N_k^n} \sum_{i:[l_n]\to[N_k]} \frac{1}{\prod\limits_{r=1}^{s} a_{N_k}^{(A_r)}} \sum_{\substack{\sigma_1\in\mathcal{S}_{N_k}^{(A_1)}\\ \sigma_2\in\mathcal{S}_{N_k}^{(A_2)}\\ \vdots \\ \sigma_s\in\mathcal{S}_{N_k}^{(A_s)}}} \prod_{a=1}^{n} \left[\left(M_{\sigma_{r_{l_{a-1}+1}}}^{\varepsilon_{l_{a-1}+1}}\right)_{i(l_{a-1}+1),i(l_{a-1}+2)} \cdot \right.$$

$$\left. \cdot \left(M_{\sigma_{r_{l_{a-1}+2}}}^{\varepsilon_{l_{a-1}+2}}\right)_{i(l_{a-1}+2),i(l_{a-1}+3)} \cdots \left(M_{\sigma_{r_{l_a}}}^{\varepsilon_{l_a}}\right)_{i(l_a),i(l_{a-1}+1)} \right] \qquad \text{[by (4.10)]}$$

$$= \frac{1}{N_k^n} \sum_{i:[l_n]\to[N_k]} \frac{1}{\prod\limits_{r=1}^{s} a_{N_k}^{(A_r)}} \operatorname{card} \left\{ (\sigma_1,\sigma_2,\dots,\sigma_s) \left| \begin{array}{l} \sigma_r\in\mathcal{S}_{N_k}^{(A_r)} \ \forall r\in[s] \text{ and} \\ \sigma_{r_a}^{\varepsilon_a}\left(i[\tau(l)]\right)=i(l) \ \forall l\in[l_n] \end{array} \right. \right\},$$

with the convention that $\sigma^* := \sigma^{-1} \ \forall \sigma \in \mathcal{S}_{N_k}$, and where $\tau \in \mathcal{S}_{l_n}$ is the permutation defined by

$$\tau = (1,2,\dots,l_1)(l_1+1,l_1+2,\dots,l_2)\cdots(l_{n-1}+1,l_{n-1}+2,\dots,l_n).$$

Next note that every function $i : [l_n] \to [N_k]$ is determined by a pair (π, f), where $\pi \in \Pi([l_n])$ is defined by

$$a \overset{\pi}{\sim} b \iff i(a) = i(b) \qquad \forall a,b \in [l_n]$$

and $f : \pi \to [N_k]$ is the (injective) function defined by

$$f\big(\pi(a)\big) = i(a) \qquad \forall a \in [l_n].$$

Let Γ be the graph having the n distinct connected components $\Gamma_{w_1}, \Gamma_{w_2}, \dots, \Gamma_{w_n}$, with each Γ_{w_a} having the vertices $1, 2, \dots, l_a - l_{a-1}$ relabeled as $l_{a-1}+1, l_{a-1}+2, \dots, l_a$. Then

$$\mathbb{E}\left(\prod_{a=1}^{n} \operatorname{tr} U_{w_a}^{(k)}\right) = \frac{1}{N_k^n} \sum_{\substack{\pi\in\Pi\big([l_n]\big)\\ f:\pi\to[N_k]\\ \text{injective}}} \frac{1}{\prod\limits_{r=1}^{s} a_{N_k}^{(A_r)}} \operatorname{card}\left\{ (\sigma_1,\sigma_2,\dots,\sigma_s) \left| \begin{array}{l} \sigma_r\in\mathcal{S}_{N_k}^{(A_r)} \ \forall r\in[s] \text{ and} \\ \sigma_{r_a}^{\varepsilon_a}\left[f\big(\pi[\tau(l)]\big)\right]=f\big(\pi(l)\big) \ \forall l\in[l_n] \end{array} \right. \right\}$$

$$= \frac{1}{N_k^n} \sum_{\pi\in\Pi\big([l_n]\big)} \sum_{\substack{f:\pi\to[N_k]\\ \text{injective}}} \frac{1}{\prod\limits_{r=1}^{s} a_{N_k}^{(A_r)}} \prod_{r=1}^{s} \operatorname{card} \mathcal{S}_{N_k}^{(A_r)}\big[f(\Gamma/\pi)(r)\big]$$

[notation as in 3.9]

$$= \frac{1}{N_k^n} \sum_{\pi\in\mathcal{C}(\Gamma)} \prod_{r=1}^{s} p_{N_k}^{(A_r)}\big[(\Gamma/\pi)(r)\big] \cdot \operatorname{card}\{f:\pi\to[N_k] \,|\, f \text{ injective}\}$$

$$= \frac{1}{N_k^n} \sum_{\pi\in\mathcal{C}(\Gamma)} \frac{N_k!}{(N_k-|\pi|)!} \prod_{r=1}^{s} p_{N_k}^{(A_r)}\big[(\Gamma/\pi)(r)\big].$$

$\square$

4.9. Corollary. *With the notation from 4.5, we have that:*

(a) For every $w_1, w_2, \ldots, w_n \in \mathbb{F}$,

$$\lim_{k\to\infty} \mathbb{E}\left(\prod_{a=1}^{n} \operatorname{tr} U_{w_a}^{(k)}\right) = \begin{cases} 1 & \text{if } w_a \approx e \ \forall a \in [n] \\ 0 & \text{otherwise.} \end{cases} \tag{4.16}$$

In addition, if each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite and it is not the case that $w_a \approx e \ \forall a \in [n]$, then

$$\mathbb{E}\left(\prod_{a=1}^{n} \operatorname{tr} U_{w_a}^{(k)}\right) = \mathrm{O}(1/N_k) \qquad \text{as } k \to \infty. \tag{4.17}$$

(b) If each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite, then for every $w_1, w_2 \in \mathbb{F}$ with $w_1 \not\approx e \not\approx w_2$, there exists $\delta > 0$ such that

$$\mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)} \cdot \operatorname{tr} U_{w_2}^{(k)}\right) = \mathrm{O}\left(1/N_k^{1+\delta}\right) \qquad \text{as } k \to \infty. \tag{4.18}$$

Proof. (a) It may be supposed that $w_a \neq e \ \forall a \in [n]$, since for every positive integer k, we have that $U_e^{(k)}$ is the constant random matrix equal to the $N_k \times N_k$ identity. Let Γ be an arbitrary graph with n connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, where each $\Gamma_a \cong \Gamma_{w_a}$. With the notation from 4.5, 2.3(a), 2.4, 2.5, 2.9, and 3.9, we have that

$$\mathbb{E}\left(\prod_{a=1}^{n} \operatorname{tr} U_{w_a}^{(k)}\right) \sim \sum_{\pi \in \mathcal{C}(\Gamma)} N_k^{|\pi|-n} \prod_{r=1}^{s} p_{N_k}^{(A_r)}[(\Gamma/\pi)(r)] \qquad \text{[by Lemma 4.8]}$$

$$\sim \sum_{\pi \in \mathcal{C}(\Gamma)} N_k^{V(\Gamma/\pi)-n} \prod_{r=1}^{s} N_k^{-E[(\Gamma/\pi)(r)]+L[(\Gamma/\pi)(r)]+\sum_{\substack{\text{loops } P \text{ of} \\ (\Gamma/\pi)(r)}} \left[\frac{l(P)}{d_r}-1\right]}$$

$$\text{[by Proposition 3.10]}$$

$$\sim \sum_{\pi \in \mathcal{C}(\Gamma)} N_k^{\chi(\Gamma/\pi)-n+\sum_{r=1}^{s} \sum_{\substack{r\text{-loops } P \\ \text{of } \Gamma/\pi}} \left[\frac{l(P)}{d_r}-1\right]} \qquad \text{as } k \to \infty.$$

For every $\pi \in \mathcal{C}(\Gamma)$, we have that $\chi(\Gamma/\pi) - n + \sum_{r=1}^{s} \sum_{\substack{r\text{-loops } P \\ \text{of } \Gamma/\pi}} \left[\frac{l(P)}{d_r} - 1\right] \leq 0$ by Proposition 2.10 and the definition of $\mathcal{C}(\Gamma)$, with equality holding iff $\pi \in \mathrm{SCon}(\Gamma)$ (notation as in 2.11) and Γ/π has n connected components, each having loop-characteristic 1; that is, iff π is of the form $\bigcup_{a=1}^{n} \pi_a$, where each $\pi_a \in \mathrm{SCon}(\Gamma_a)$ and $\chi(\Gamma_a/\pi_a) = 1$. Thus

$$\lim_{k\to\infty} \mathbb{E}\left(\prod_{a=1}^{n} \operatorname{tr} U_{w_a}^{(k)}\right) = \prod_{a=1}^{n} \operatorname{card}\left\{\pi \in \mathrm{SCon}(\Gamma_a) \,|\, \chi(\Gamma_a/\pi) = 1\right\} = \begin{cases} 1 & \text{if } w_a \approx e \ \forall a \in [n] \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{[by Proposition 2.17]}$$

In addition, if each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite and it is not the case that $w_a \approx e \; \forall a \in [n]$, then $\chi(\Gamma/\pi) - n + \sum_{r=1}^{s} \sum_{\substack{r\text{-loops } P \\ \text{of } \Gamma/\pi}} \left[\frac{l(P)}{d_r} - 1 \right]$ is

a negative integer for every $\pi \in \mathcal{C}(\Gamma)$, so it follows that

$$\mathbb{E}\left(\prod_{a=1}^{n} \operatorname{tr} U_{w_a}^{(k)} \right) = \mathrm{O}(1/N_k) \qquad \text{as } k \to \infty.$$

(b) Suppose that each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite, and let $w_1, w_2 \in \mathbb{F}$ be such that $w_1 \not\approx e \not\approx w_2$. Let Γ be an arbitrary graph with two connected components Γ_1 and Γ_2, where $\Gamma_1 \cong \Gamma_{w_1}$ and $\Gamma_2 \cong \Gamma_{w_2}$. Exactly as in the proof of part (a), we have that

$$\mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)} \cdot \operatorname{tr} U_{w_2}^{(k)} \right) \sim \sum_{\pi \in \mathcal{C}(\Gamma)} N_k^{\chi(\Gamma/\pi) - 2 + \sum_{r=1}^{s} \sum_{\substack{r\text{-loops } P \\ \text{of } \Gamma/\pi}} \left[\frac{l(P)}{d_r} - 1 \right]} \qquad \text{as } k \to \infty,$$

so it suffices to show that $\chi(\Gamma/\pi) - 2 + \sum_{r=1}^{s} \sum_{\substack{r\text{-loops } P \\ \text{of } \Gamma/\pi}} \left[\frac{l(P)}{d_r} - 1 \right] < -1$ for every $\pi \in \mathcal{C}(\Gamma)$.

Suppose first that $\pi \in \mathcal{C}(\Gamma)$ is of the form $\pi_1 \cup \pi_2$ for some partitions π_1 and π_2 of the vertex sets of Γ_1 and Γ_2, respectively. Then Γ/π consists of the two connected components Γ_1/π_1 and Γ_2/π_2, so

$$\chi(\Gamma/\pi) - 2 + \sum_{r=1}^{s} \sum_{\substack{r\text{-loops } P \\ \text{of } \Gamma/\pi}} \left[\frac{l(P)}{d_r} - 1 \right]$$

$$= \left[\chi(\Gamma_1/\pi_1) - 1 + \sum_{r=1}^{s} \sum_{\substack{r\text{-loops } P \\ \text{of } \Gamma_1/\pi_1}} \left[\frac{l(P)}{d_r} - 1 \right] \right] + \left[\chi(\Gamma_2/\pi_2) - 1 + \sum_{r=1}^{s} \sum_{\substack{r\text{-loops } P \\ \text{of } \Gamma_2/\pi_2}} \left[\frac{l(P)}{d_r} - 1 \right] \right].$$

As in the proof of part (a), each of the two summands is a negative integer, so it follows that $\chi(\Gamma/\pi) - 2 + \sum_{r=1}^{s} \sum_{\substack{r\text{-loops } P \\ \text{of } \Gamma/\pi}} \left[\frac{l(P)}{d_r} - 1 \right] \le -2$.

Finally, suppose that $\pi \in \mathcal{C}(\Gamma)$ is not of the form $\pi_1 \cup \pi_2$ for some partitions π_1 and π_2 of the vertex sets of Γ_1 and Γ_2, respectively. Then Γ/π is connected, so to show that $\chi(\Gamma/\pi) - 2 + \sum_{r=1}^{s} \sum_{\substack{r\text{-loops } P \\ \text{of } \Gamma/\pi}} \left[\frac{l(P)}{d_r} - 1 \right] < -1$, it suffices to show that either Γ/π is not strongly admissible or $\chi(\Gamma/\pi) < 1$. Suppose that Γ/π is strongly admissible and that $\chi(\Gamma/\pi) = 1$, and let π_1 be the restriction of π to the vertices of Γ_1, as in 2.2(a). Then Γ_1/π_1 is isomorphic to a subgraph of Γ/π, so Γ_1/π_1 is strongly admissible and $\chi(\Gamma_1/\pi_1) = 1$ by Remarks 2.12. Thus $\pi_1 \in \mathrm{SCon}(\Gamma_1)$, which contradicts Proposition 2.17 (since $w_1 \not\approx e$). $\qquad \square$

Proof of Theorem 4.6. (a) It suffices to prove (4.11), and this follows directly from (4.9) and Corollary 4.9(a).

(b) It suffices to prove (4.14). Suppose that each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite, and let $w_1, w_2 \in \mathbb{F} \setminus \{e\}$.

If $w_1 \approx e$, then for every positive integer k, we have that $U_{w_1}^{(k)}$ is the constant random matrix equal to the $N_k \times N_k$ identity [see the remark at the end of 4.5(c)], so

$$\mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)} \cdot \operatorname{tr} U_{w_2}^{(k)}\right) - \mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)}\right) \cdot \mathbb{E}\left(\operatorname{tr} U_{w_2}^{(k)}\right) = \mathbb{E}\left(\operatorname{tr} U_{w_2}^{(k)}\right) - \mathbb{E}\left(\operatorname{tr} U_{w_2}^{(k)}\right) = 0.$$

Similarly, if $w_2 \approx e$, then $\mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)} \cdot \operatorname{tr} U_{w_2}^{(k)}\right) - \mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)}\right) \cdot \mathbb{E}\left(\operatorname{tr} U_{w_2}^{(k)}\right) = 0.$

Finally, if $w_1 \not\approx e \not\approx w_2$, then

$$\mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)}\right) \cdot \mathbb{E}\left(\operatorname{tr} U_{w_2}^{(k)}\right) = \mathrm{O}\left(1/N_k^2\right) \qquad \text{as } k \to \infty$$

and there exists $\delta > 0$ such that

$$\mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)} \cdot \operatorname{tr} U_{w_2}^{(k)}\right) = \mathrm{O}\left(1/N_k^{1+\delta}\right) \qquad \text{as } k \to \infty$$

by Corollary 4.9, so $\sum_{k=1}^{\infty} \left[\mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)} \cdot \operatorname{tr} U_{w_2}^{(k)}\right) - \mathbb{E}\left(\operatorname{tr} U_{w_1}^{(k)}\right) \cdot \mathbb{E}\left(\operatorname{tr} U_{w_2}^{(k)}\right)\right] < \infty.$ $\qquad\square$

4.10. Asymptotic freeness from Gaussian matrices. Within the framework of Theorem 4.6, the family $\left(U_r^{(k)}\right)_{r=1}^{s}$ is also asymptotically $*$-free from either an independent family of square complex Gaussian matrices, or from an independent family of complex Wishart matrices. More concretely, let $(f_{r;i,j})_{\substack{r\in[s] \\ i,j\geq 1}}$ be an independent family of complex standard Gaussian random variables over $(\Omega, \mathcal{F}, P)$; in other words, $(\operatorname{Re} f_{r;i,j}, \operatorname{Im} f_{r;i,j})_{\substack{r\in[s] \\ i,j\geq 1}}$ is an independent family of real Gaussian random variables over $(\Omega, \mathcal{F}, P)$ with mean 0 and variance 1/2. For every $M, N \geq 1$, define random $M \times N$ matrices $G_1^{(M,N)}, G_2^{(M,N)}, \ldots, G_s^{(M,N)}$ by

$$\left(G_r^{(M,N)}\right)_{i,j} = f_{r;i,j} \qquad \forall i \in [M], \forall j \in [N]. \tag{4.19}$$

Fix a sequence $(M_k)_{k=1}^{\infty}$ of integers with the ratio M_k/N_k converging to some real number c and for every $r \in [s]$ and $k \geq 1$, let $G_r^{(k)} := \frac{1}{\sqrt{N_k}} G_r^{(N_k, N_k)}$ and $W_r^{(k)} := \frac{1}{N}\left(G_r^{(M_k, N_k)}\right)^* G_r^{(M_k, N_k)}$. (The random $N_k \times N_k$ matrices $W_r^{(k)}$ are known as *complex Wishart matrices*.) Then an almost identical proof to that of Theorem 3.1 of [4] shows that:

(1) If the family $\left(U_r^{(k)}, G_r^{(k)}\right)_{r=1}^{s}$ is independent for every $k \geq 1$, then $\left(U_r^{(k)}, G_r^{(k)}\right)_{r=1}^{s} \xrightarrow[k\to\infty]{*\text{-distr.}}$ $(u_r, g_r)_{r=1}^{s}$, where each g_r is a *standard circular* random variable in $(\mathcal{A}, \varphi)$ (see [4] for the definition) and the family $(u_r, g_r)_{r=1}^{s}$ is $*$-free.

(2) If the family $\left(U_r^{(k)}, W_r^{(k)}\right)_{r=1}^{s}$ is independent for every $k \geq 1$, then $\left(U_r^{(k)}, W_r^{(k)}\right)_{r=1}^{s} \xrightarrow[k\to\infty]{*\text{-distr.}}$ $(u_r, w_r)_{r=1}^{s}$, where each w_r is a *free Poisson* random variable of parameter c in $(\mathcal{A}, \varphi)$ (see [4] for the definition) and the family $(u_r, w_r)_{r=1}^{s}$ is $*$-free.

In both cases, if each of the sets $A_1, A_2, \ldots, A_s$ either consists of a single positive integer or is infinite, then the convergence in $*$-distribution actually holds almost surely.

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